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Continuity, End Behavior, and Limits

Unit 1 Lesson 3

Students will be able to:

Interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Key Vocabulary:

Discontinuity,
A limit,
End Behavior

The graph of **a continuous function** has no breaks, holes, or gaps. You can trace the graph of a continuous function without lifting your pencil.

One condition for a function $f(x)$ to be continuous at $x = c$ is that the function must approach a unique function value as x -values approach c from the left and right sides.

The concept of approaching a value without necessarily ever reaching it is called **a limit.**

If the value of $f(x)$ approaches a unique value L as x approaches c from each side, then the limit of $f(x)$ as x approaches c is L.

$$\lim_{x \to c} f(x) = L$$

Functions that are not continuous are discontinuous.

Graphs that are discontinuous can exhibit:

- **Infinite discontinuity**

A function has an infinite discontinuity at $x = c$, if the function value increases or decreases indefinitely as x approaches c from the left and right.

Functions that are not continuous are discontinuous. Graphs that are discontinuous can exhibit:

- **Jump discontinuity**

A function has a jump discontinuity at $x = c$ if the limits of the function as x approaches c from the left and right exist but have two distinct values.

Functions that are not continuous are discontinuous.

Graphs that are discontinuous can exhibit:

- **Removable discontinuity, also called point discontinuity**

Function has a removable discontinuity if the function is continuous everywhere except for a hole at $x = c$.

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Continuity Test

A function $\boldsymbol{f(x)}$ is continuous at $\boldsymbol{x = c}$, if it satisfies the following conditions:

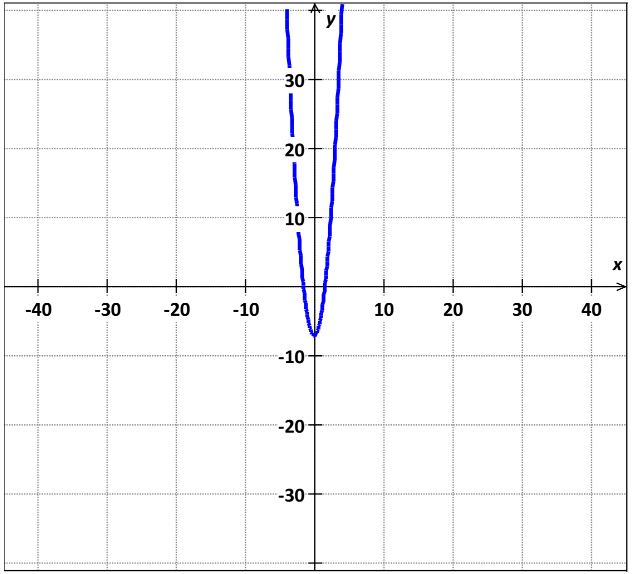
1. $\boldsymbol{f(x)}$ is defined at **c.** $\boldsymbol{f(c)}$exists.

2. $\boldsymbol{f(x)}$ approaches the same function value to the left and right of $\boldsymbol{c}$. $\lim\limits_{x \to c} f(x)$ ***exists***

3. The function value that $\boldsymbol{f(x)}$approaches from each side of $\boldsymbol{c}$ is $\boldsymbol{f(c)}$. $\lim\limits_{x \to c} f(x) = f(c)$

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

a. $f(x) = 3x^2 + x - 7 \quad at \;\; x = \;1$



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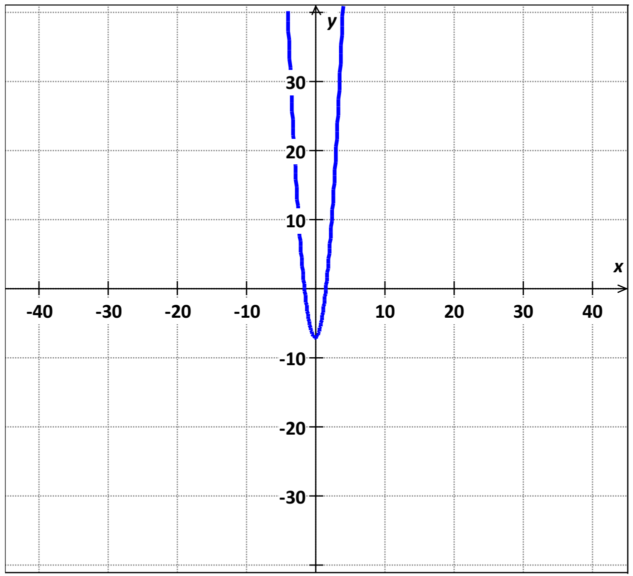

$$f(1) = 3 * 1^2 + 1 - 7 = -3$$

$$f(1) \; exists$$

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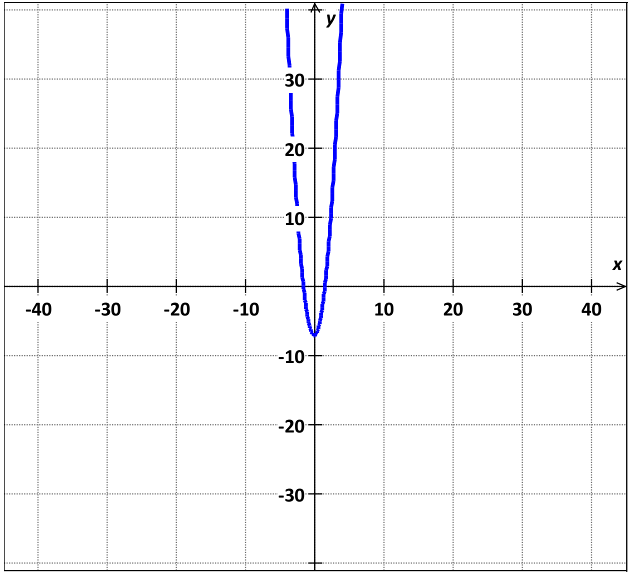

$x \to 1^- \quad y \to -3$

x	0.9	0.99	0.999
$f(x)$	-3.67	-3.0697	-3.006997

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

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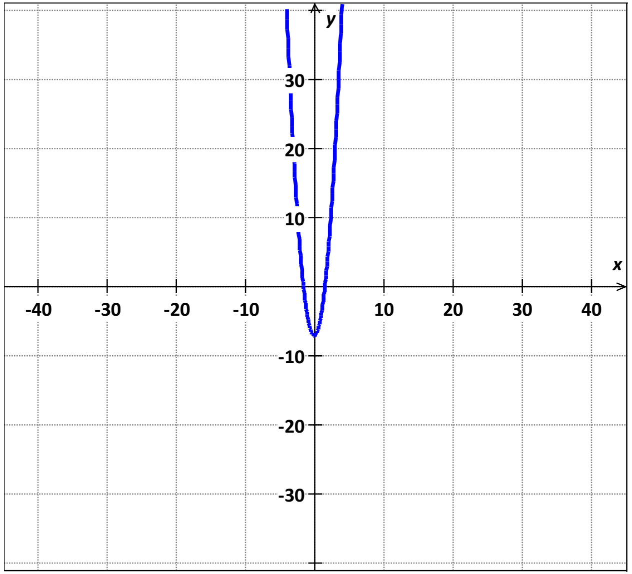
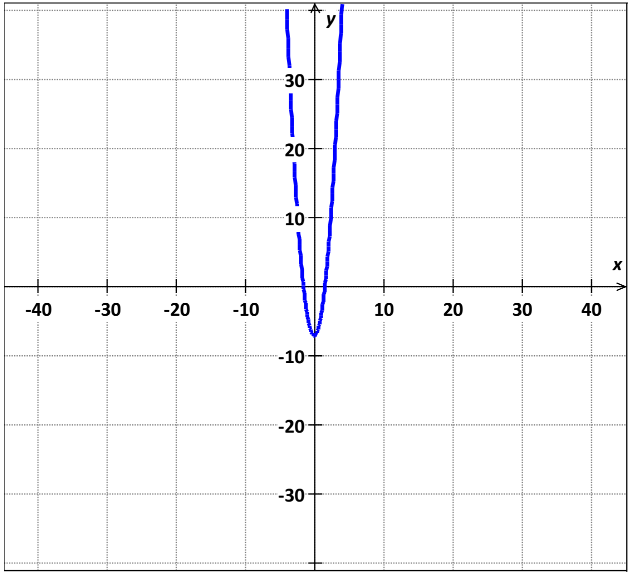


$x \to 1^+ \;\; y \to -3$

x	1.1	1.01	1.001
$f(x)$	-2.27	-2.9297	-2.992997

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

a. $f(x) = 3x^2 + x - 7 \quad at\ \ x = 1$



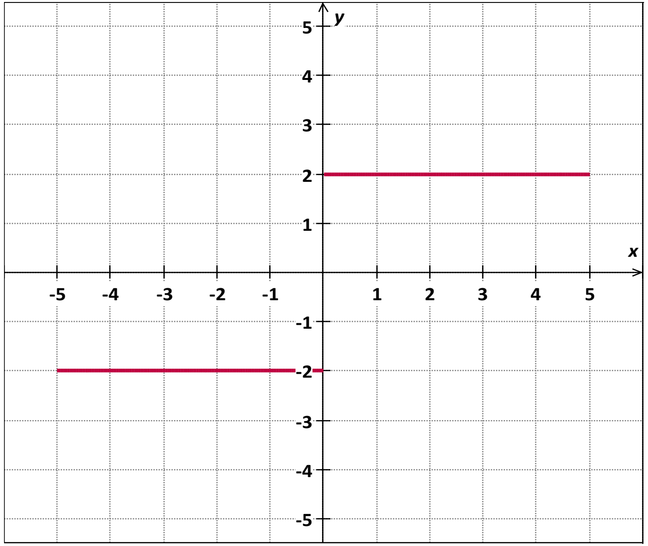
$f(1) = -3\ \ and\ \ y \rightarrow -3$
$from\ both\ side\ of\ x = 1$

$$\lim_{x \to 1} 3x^2 + x - 7 = f(1)$$

$f(x) = 3x^2 + x - 7$
$is\ continuous\ at\ x = 1$

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

b. $f(x) = \dfrac{|2x|}{x} \quad at \quad x = 0$



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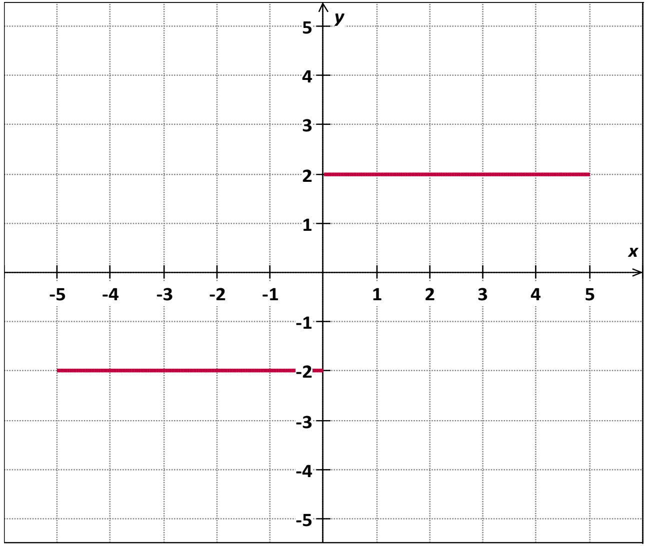

$$f(0) = \frac{|2 * 0|}{0} = \frac{0}{0}$$

$The\ function\ is\ undefined\ at$
$x = 0$

Sample Problem 1: Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.

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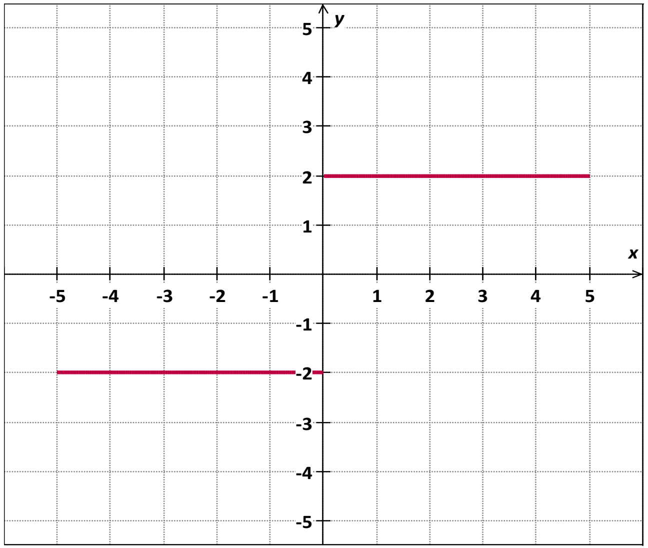

$x \to 0^- \quad y \to -2$

x	-0.1	-0.01	-0.001
$f(x)$	-2	-2	-2

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

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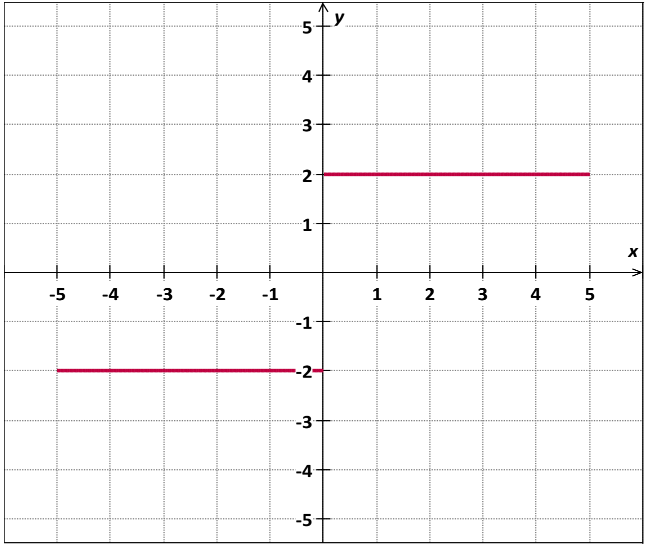
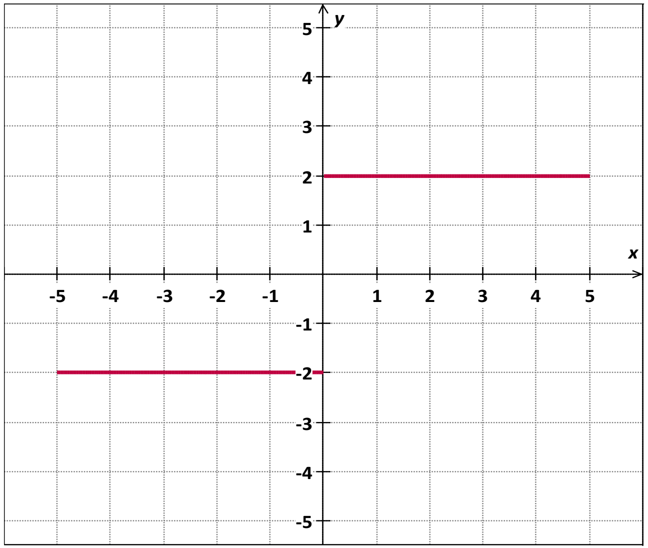


$x \to 0^+ \ \ y \to 2$

x	0.1	0.01	0.001
$f(x)$	2	2	2

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

b. $f(x) = \dfrac{|2x|}{x} \quad at \quad x = 0$



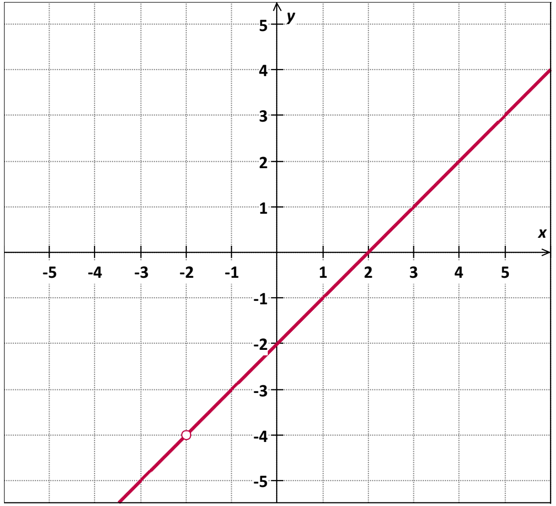
$f(x) = \dfrac{|2x|}{x}$

has jump discontinuity at $x = 0$

since y *values are* 2 *and* -2 *on opposite sides of* $x = 0$

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

c. $f(x) = \dfrac{x^2 - 4}{x + 2} \quad at \quad x = -2$



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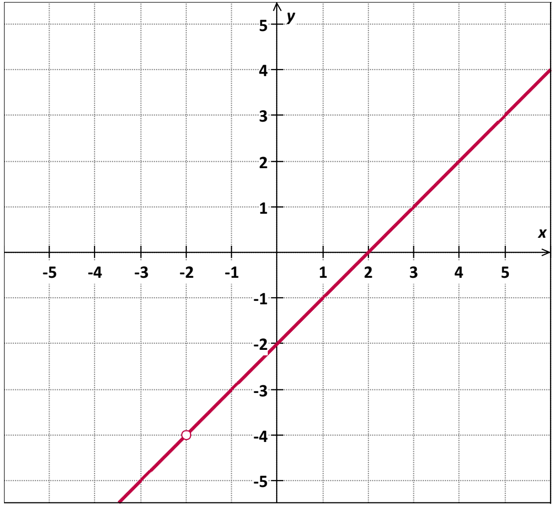

$$f(-2) = \frac{(-2)^2 - 4}{-2 + 2} = \frac{0}{0}$$

$f(x)$ *is undefined at* $x = -2$

$$f(x) = \frac{x^2 - 4}{x + 2}$$

is discontinuous at $x = -2$

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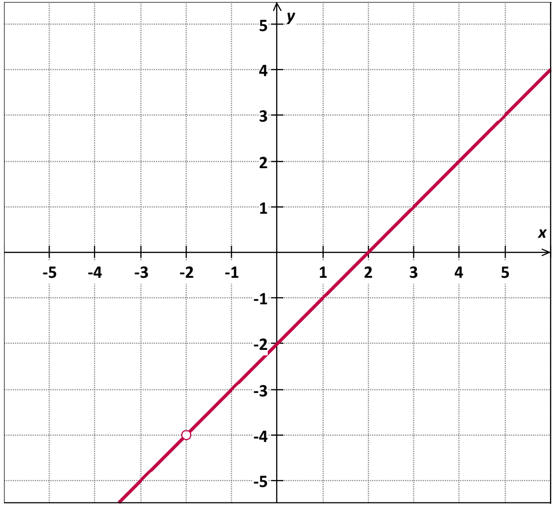

$x \to -2^- \quad y \to -4$

x	-2.1	-2.01	-2.001
$f(x)$	-4.1	-4.01	-4.001

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

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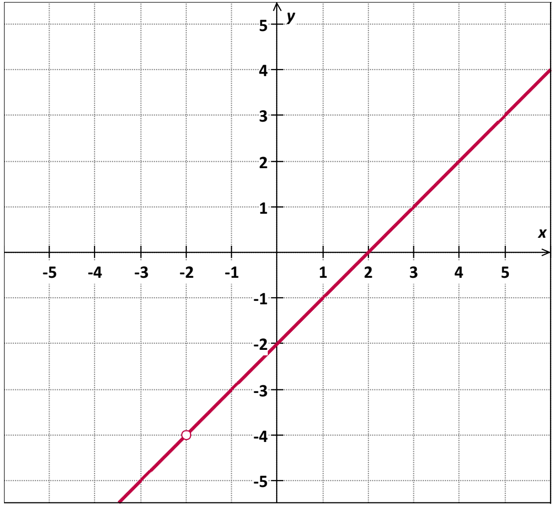

$x \to -2^+ \;\; y \to -4$

x	-1.9	-1.99	-1.999
$f(x)$	-3.9	-3.99	-3.999

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

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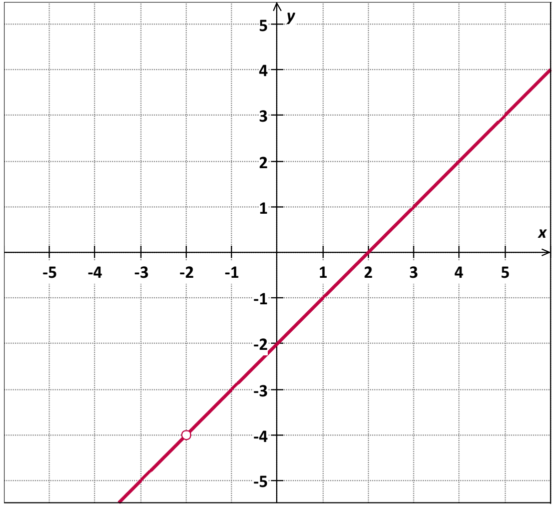


$$f(x) = \frac{x^2 - 4}{x + 2}$$

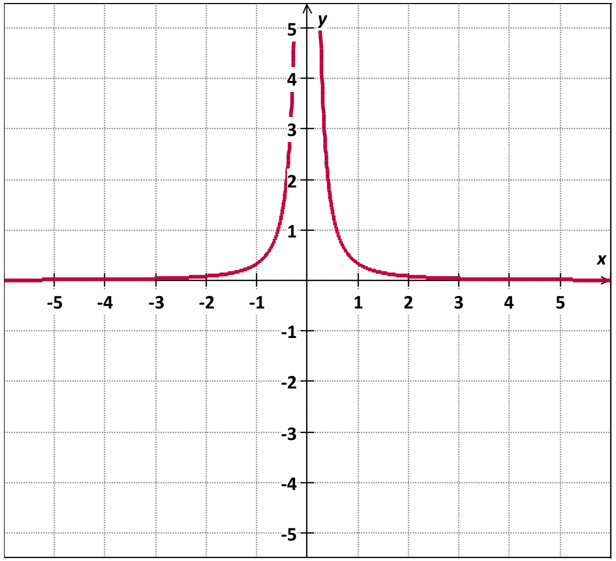
has point discontinuity at $x = -2$

since y *value is* -4

on opposite sides of $x = -2$

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

d. $f(x) = \dfrac{1}{3x^2} \quad at \quad x = 0$



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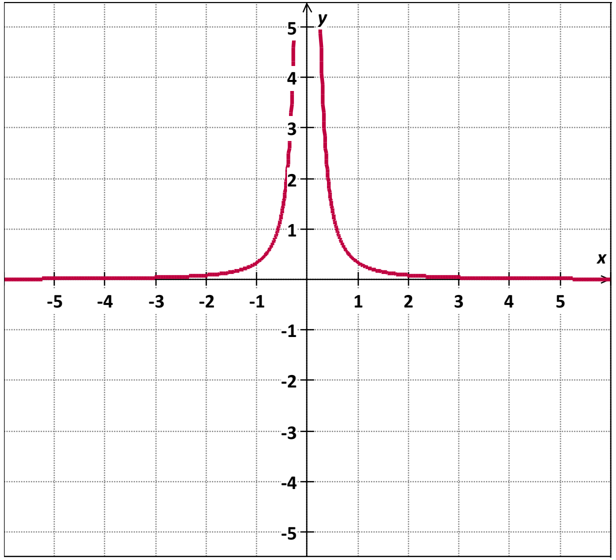

$$f(0) = \frac{1}{3 * 0^2} = \infty$$

$f(x)$ *is undefined at* $x = 0$

$$f(x) = \frac{1}{3x^2}$$

is discontinuous at $x = 0$

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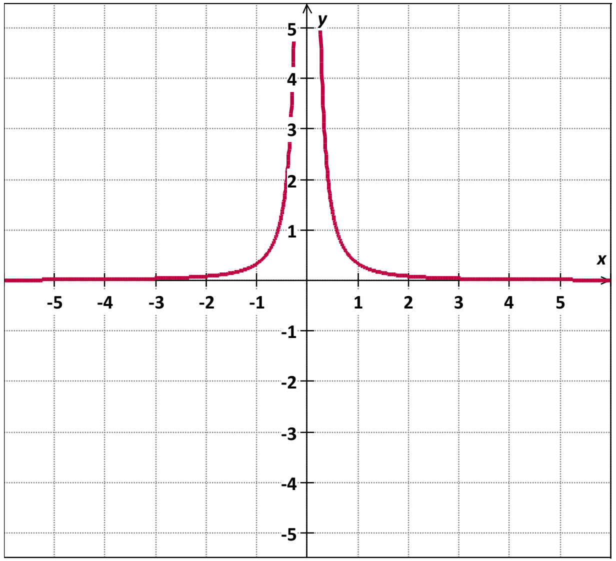

$x \to 0^- \quad y \to +\infty$

x	-0.1	-0.01	-0.001
$f(x)$	33.33	$3,333.33$	$333,333.33$

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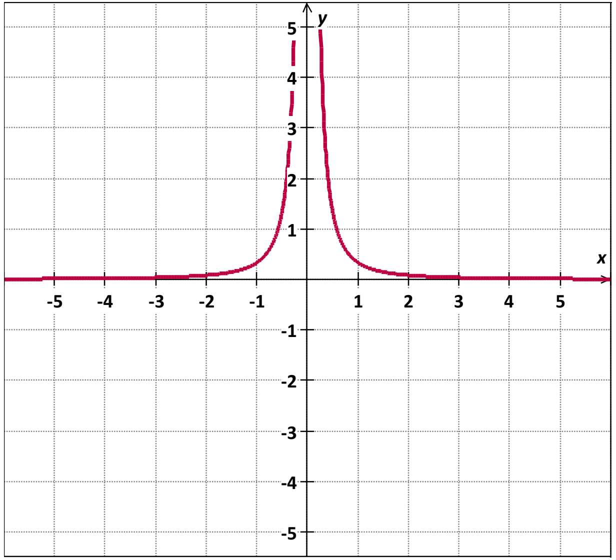

$x \to 0^+ \qquad y \to +\infty$

x	0.1	0.01	0.001
$f(x)$	33.33	$3,333.33$	$333,333.33$

Sample Problem 1: **Determine whether each function is continuous at the given x -values. Justify using the continuity test. If discontinuous, identify the type of discontinuity.**

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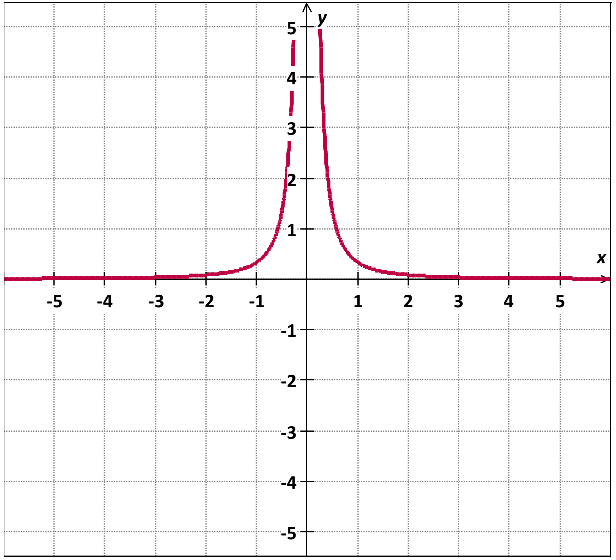


$f(x) = \dfrac{1}{3x^2}$

$has\ infinity\ discontinuity$

$at \quad x = 0$

$since \quad y \ \ value\ is + \infty$

$when \ \ x \to 0$

Intermediate Value Theorem

If $\boldsymbol{f(x)}$ is a continuous function and $\boldsymbol{a < b}$ and there is a value $\boldsymbol{n}$ such that $\boldsymbol{n}$ is between $\boldsymbol{f(a)}$ and $\boldsymbol{f(b)}$, then there is a number $\boldsymbol{c}$, such that $\boldsymbol{a < c < b}$ and $\boldsymbol{f(c) = n}$

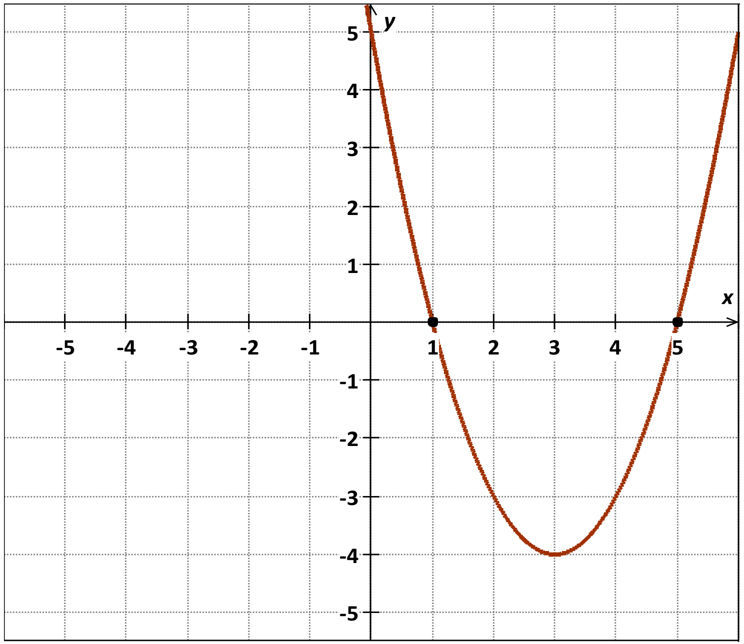
The Location Principle

If $f(x)$ is a continuous function and $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c, such that $a < c < b$ and $f(c) = 0$.

That is, there is a zero between a and b.

Sample Problem 2: **Determine between which consecutive integers the real zeros of function are located on the given interval.**

a. $f(x) = (x-3)^2 - 4$ $[0, 6]$



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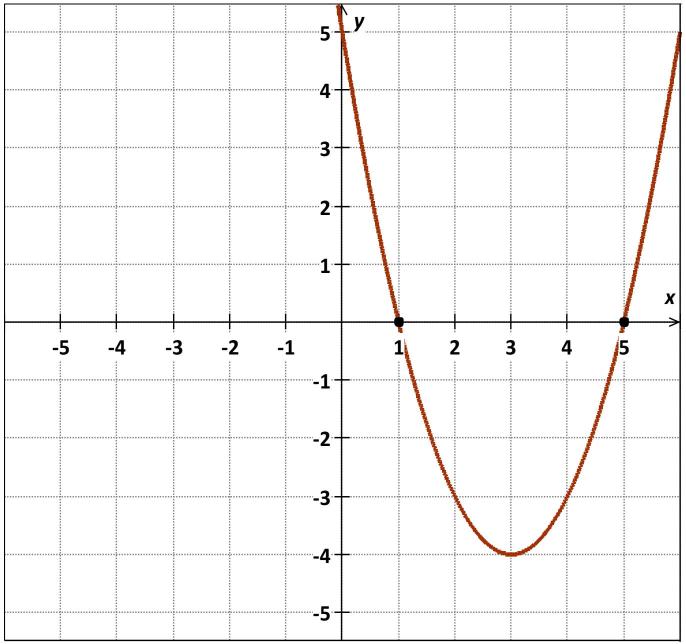


x	0	1	2	3	4	5	6
y	5	0	-3	-4	-3	0	5

$f(0)$ is positive and $f(2)$ is negative,

$f(x)$ *change sign in* $0 \leq x \leq 2$

$f(4)$ is negative and $f(6)$ is positive,

$f(x)$ *change sign in* $4 \leq x \leq 6$

$f(x)$ has zeros in intervals:

$0 \leq x \leq 2$ *and* $4 \leq x \leq 6$

End Behavior

The end behavior of a function describes what the y -values do as $|x|$ becomes greater and greater.

When x becomes greater and greater, we say that x approaches infinity, and we write $x \to +\infty$.

When x becomes more and more negative, we say that x approaches negative infinity, and we write $x \to -\infty$.

The same notation can also be used with y or $f(x)$ and with real numbers instead of infinity.

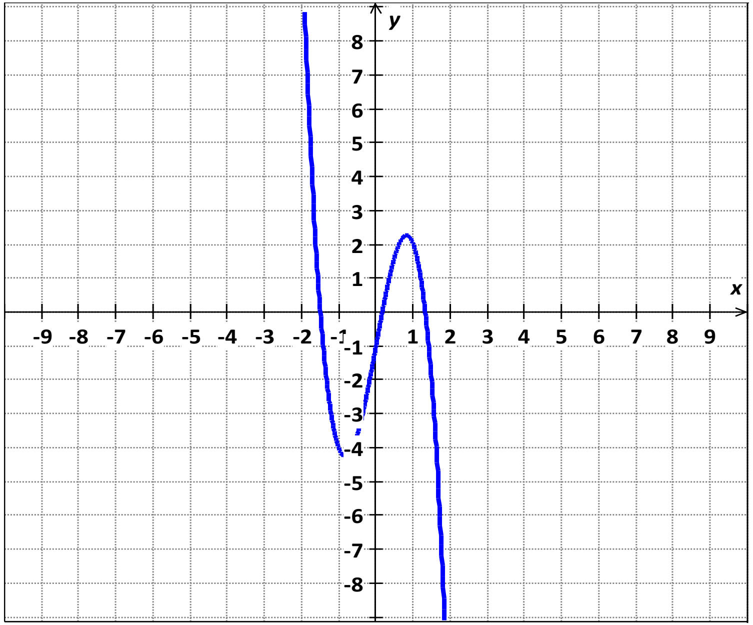
Left - End Behavior (as x becomes more and more negative): $\lim_{x \to -\infty} f(x)$

Right - End Behavior (as x becomes more and more positive): $\lim_{x \to +\infty} f(x)$

The $f(x)$ values may approach negative infinity, positive infinity, or a specific value.

Sample Problem 3: **Use the graph of each function to describe its end behavior. Support the conjecture numerically.**

a. $f(x) = -3x^3 + 6x - 1$



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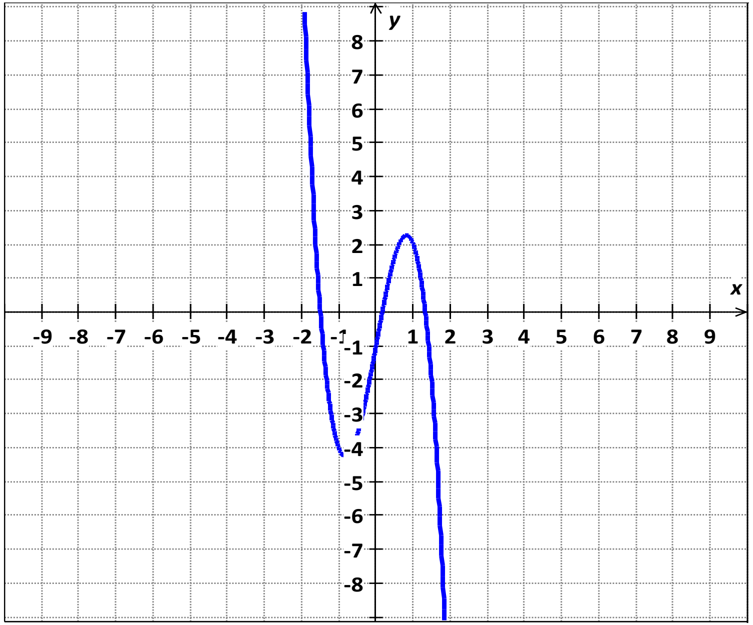


From the graph, it appears that:

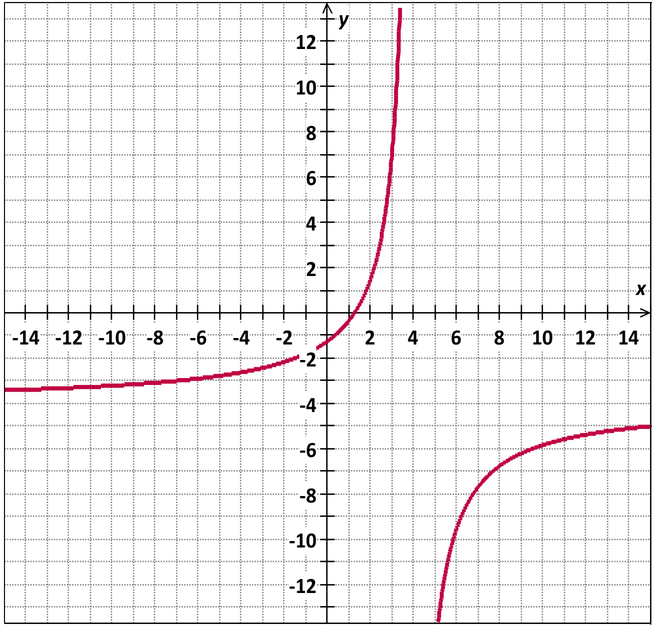
$$f(x) \to \infty \text{ as } x \to -\infty$$
$$f(x) \to -\infty \text{ as } x \to \infty$$

The table supports this conjecture.

x	-10^4	-10^3	0	10^3	10^4
y	$3 * 10^{12}$	$3 * 10^9$	-1	$-3 * 10^9$	$-3 * 10^{12}$

Sample Problem 3: **Use the graph of each function to describe its end behavior. Support the conjecture numerically.**

b. $f(x) = \dfrac{4x - 5}{4 - x}$



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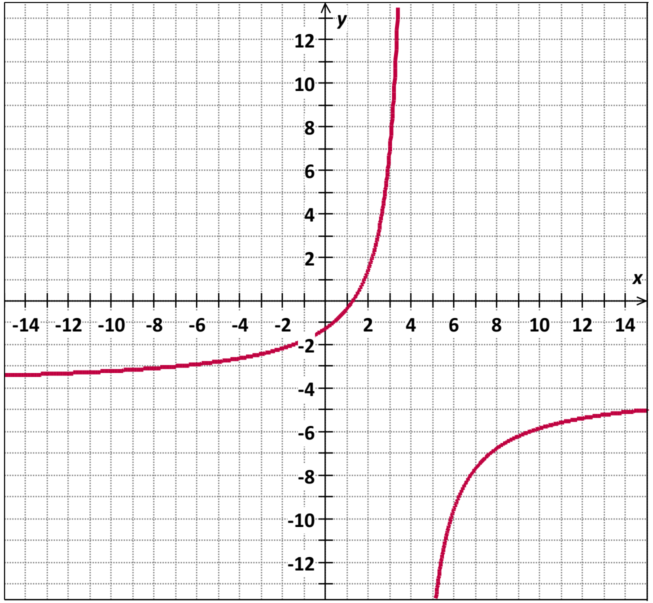


From the graph, it appears that:

$$f(x) \to -4 \text{ as } x \to -\infty$$
$$f(x) \to -4 \text{ as } x \to \infty$$

The table supports this conjecture.

x	-10^4	-10^3	0	10^3	10^4
y	-3.9989	-3.9890	-1.25	-4.001	-4.0011

Increasing, Decreasing, and Constant Functions

A function $\boldsymbol{f}$ is increasing on an interval $\boldsymbol{I}$ if and only if for every $\boldsymbol{a}$ and $\boldsymbol{b}$ contained in $\boldsymbol{I}$, $(\boldsymbol{a}) < \boldsymbol{f}(\boldsymbol{b})$, whenever $\boldsymbol{a} < \boldsymbol{b}$.

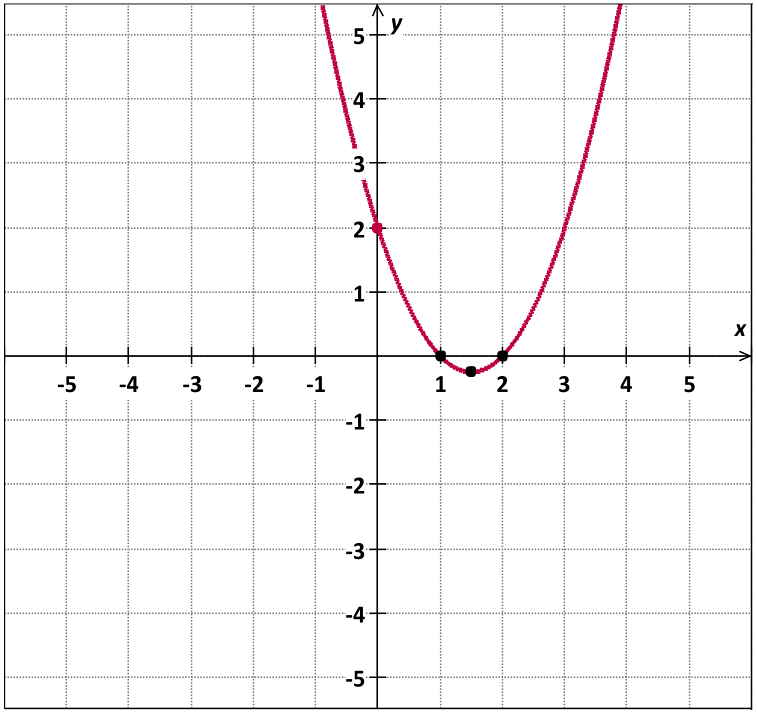
A function $\boldsymbol{f}$ is decreasing on an interval $\boldsymbol{I}$ if and only if for every $\boldsymbol{a}$ and $\boldsymbol{b}$ contained in $\boldsymbol{I}$, $\boldsymbol{f}(\boldsymbol{a}) > \boldsymbol{f}(\boldsymbol{b})$ whenever $\boldsymbol{a} < \boldsymbol{b}$.

A function $\boldsymbol{f}$ remains constant on an interval $\boldsymbol{I}$ if and only if for every $\boldsymbol{a}$ and $\boldsymbol{b}$ contained in $\boldsymbol{I}$, $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{f}(\boldsymbol{b})$ whenever $\boldsymbol{a} < \boldsymbol{b}$.

Points in the domain of a function where the function changes from increasing to decreasing or from decreasing to increasing are called **critical points.**

Sample Problem 4: Determine the interval(s) on which the function is increasing and the interval(s) on which the function is decreasing.

a. $f(x) = x^2 - 3x + 2$



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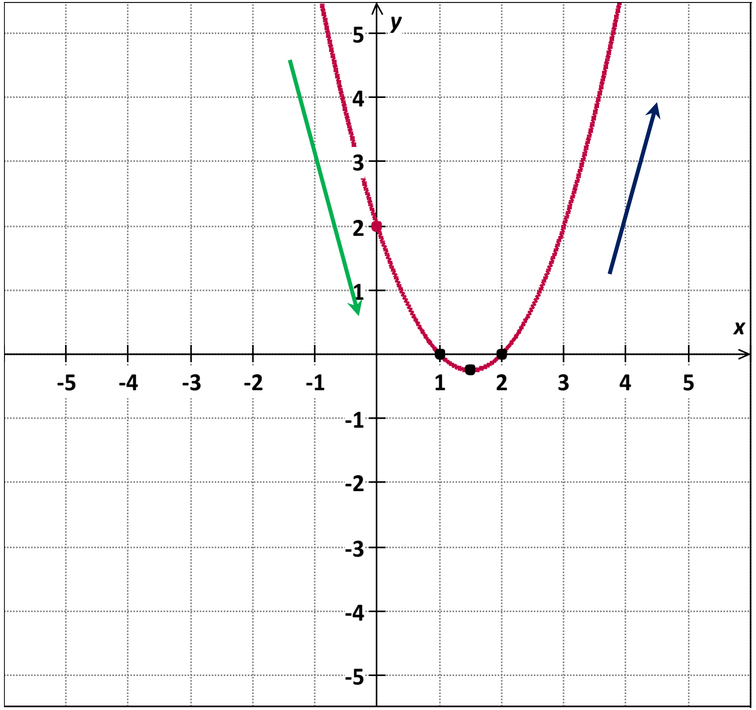


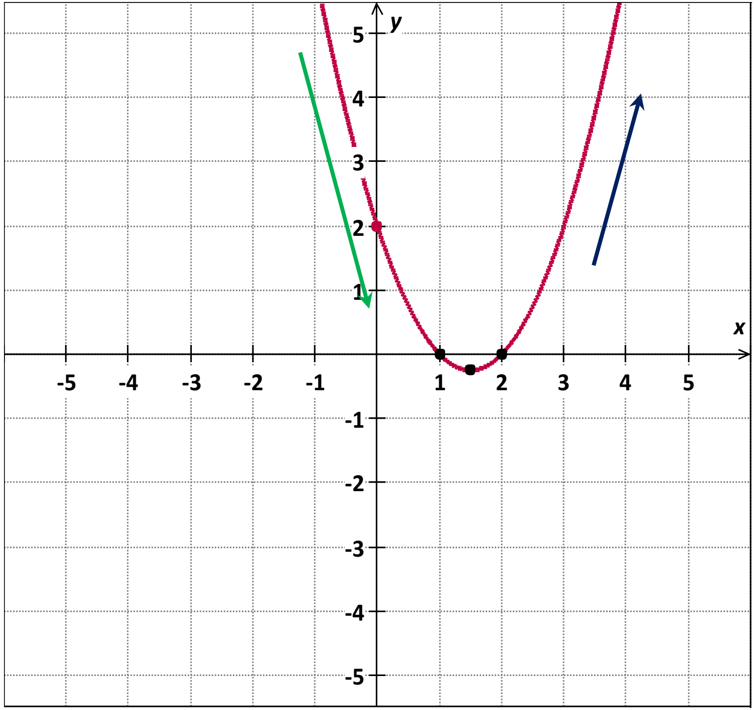
From the graph, it appears that:

A function $x^2 - 3x + 2$ is decreasing for $x < 1.5$

A function $x^2 - 3x + 2$ is increasing for $x > 1.5$

Sample Problem 4: Determine the interval(s) on which the function is increasing and the interval(s) on which the function is decreasing.

a. $f(x) = x^2 - 3x + 2$

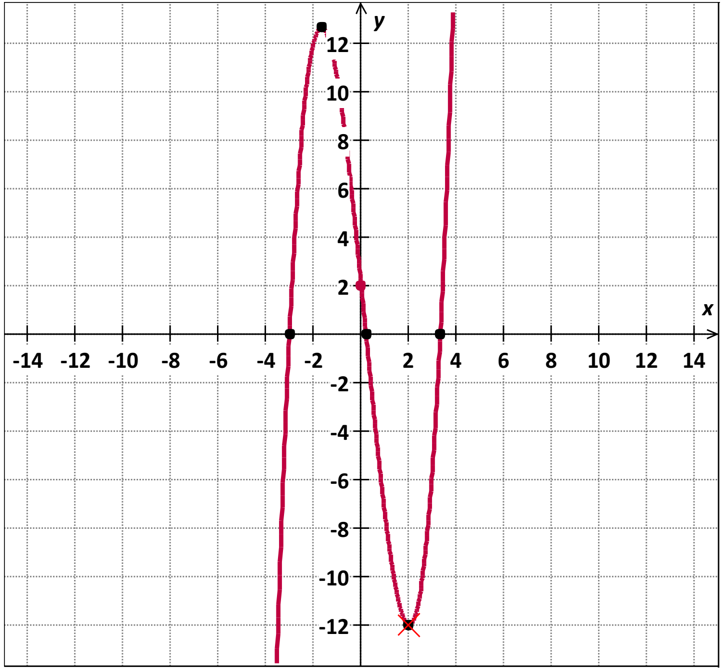


The table supports this conjecture.

x	-1	0	1	1.5	2	3
y	6	2	0	-0.25	-5.5	2

Sample Problem 4: **Determine the interval(s) on which the function is increasing and the interval(s) on which the function is decreasing.**

b. $$f(x) = x^3 - \frac{1}{2}x^2 - 10x + 2$$



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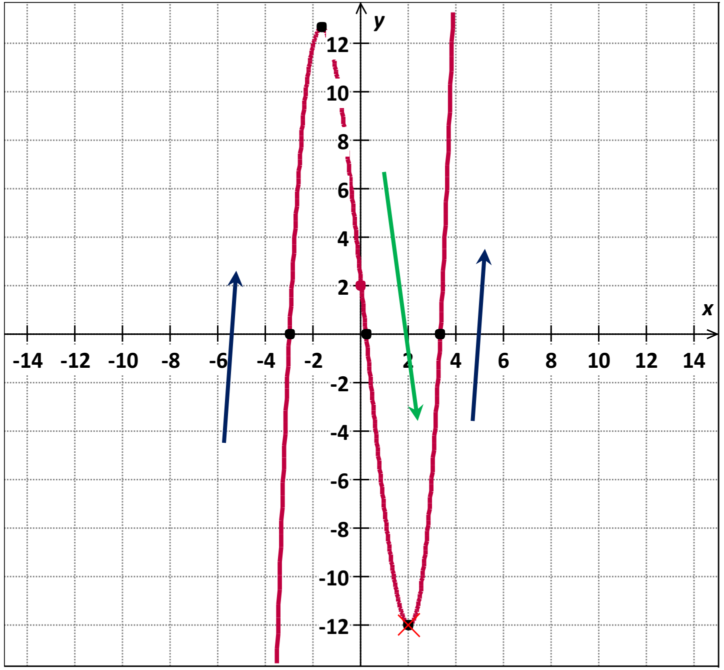


From the graph, it appears that:

A function $x^3 - \frac{1}{2}x^2 - 10x + 2$ is increasing:

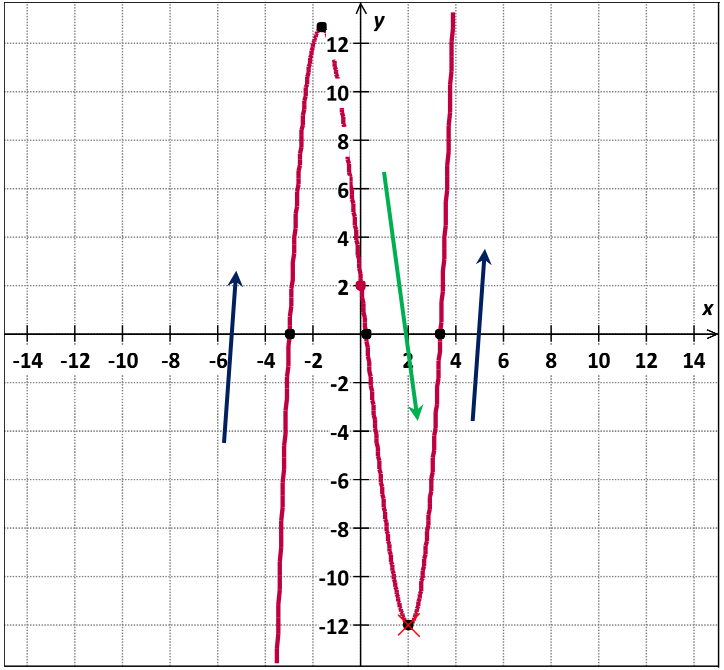
$x < -1.66 \;\; and \;\; x > 2$

A function $x^3 - \frac{1}{2}x^2 - 10x + 2$ is decreasing:

$-1.66 < x < 2$

Sample Problem 4: Determine the interval(s) on which the function is increasing and the interval(s) on which the function is decreasing.

b. $f(x) = x^3 - \frac{1}{2}x^2 - 10x + 2$



The table supports this conjecture.

x	-2	-1.66	-1	0	2	3
y	12	12.65	10.5	2	-12	-5.5